

GROUP THEORY 2024 - 25, EXERCISE SHEET 12

Exercise 1. (hard) *To always do in every course!*

Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) Let

$$1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$$

be a short exact sequence of groups such that F is a free group. Show that the sequence splits.

Exercise 3. (easy) Let F be a free group on a set S , where $|S| \geq 2$. Show that F is torsion free and has trivial centre.

Exercise 4. (easy) Let X and Y be disjoint sets. Let N be the normal subgroup of $F_{X \cup Y}$ generated by Y . Show that $F_{X \cup Y}/N \cong F_X$.

Exercise 5. *The Quaternion Group* (medium)

Consider the group:

$$Q_8 := \langle i, j, k \mid i^4 = 1, i^2 = j^2 = k^2 = ijk \rangle.$$

The group Q_8 is called the Quaternion group.

- (1) Define the element $-1 := i^2$. Show that -1 is in the the centre of Q_8 .
- (2) Show that Q_8 is a finite group of order 8.
- (3) Find the centre of Q_8 .
- (4) Show that all subgroups of Q_8 are normal subgroups.

Exercise 6. (medium)

Compute the abelianisations of the following groups:

- (1) A_5
- (2) A_4
- (3) S_n for $n \geq 5$.
- (4) F_S where $|S| = 2$.
- (5) $\langle a, b \mid a^2b^3, a^4b^5 \rangle$.

Exercise 7. (Hard)

- (1) Show that $\langle a, b \mid a^2 = 1, b^2 = 1, (ab)^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- (2) Show that $\langle a, b \mid a^3 = 1, b^3 = 1, (ab)^2 = 1 \rangle \cong A_4$.
- (3) Prove that A_5 can be generated by an element of order 5 and an element of order 2.
- (4) Prove that F_S is not solvable whenever $|S| \geq 2$.